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# Quantization on $Z_M$ and coherent states over $Z_M \times Z_M$

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**Abstract.** Finite-dimensional quantum mechanics (quantum mechanics on finite discrete space $Z_M$—the cyclic group of order $M$) is developed further: in analogy with the usual harmonic oscillator coherent states, an overcomplete family of coherent states over the phase space $Z_M \times Z_M$ is constructed and their properties are determined.

## 1. Introduction

Coherent states belong to the most important tools in numerous applications of the quantum theory. They found many various applications in quantum optics, quantum field theory, quantum statistical mechanics [1, 2]. Our papers [3, 4] were devoted to the basic notions of finite-dimensional quantum mechanics on configuration space $Z_M$, where $Z_M$ is the cyclic group of order $M$. In [3], our approach was based on Mackey's irreducible system of impritivity. Thereby, we were able to extend Schwinger's treatment to an arbitrary Weyl system in finite-dimensional Hilbert spaces. In order to generalize the notion of coherent states to this framework, i.e. over the phase space $Z_M \times Z_M$, we make consequent use of the analogy with the usual harmonic oscillator coherent states.

Finite-dimensional quantum mechanics was first developed by Weyl [5] in connection with the investigation of Abelian groups of rotations of projective spaces. Weyl constructed a set of unitary operators in a Hilbert space of dimension $M$ in complete analogy with the Weyl system over the Euclidean configuration space. Recently, in quantum optics, discrete phase space appeared in connection with the quantum description of phase conjugated to number operators [6], with the introduction of discrete quasi-distributions [7], and with rotation angle—angular momentum variables [8]. The group theoretical point of view of canonical transformations on a finite phase space was studied in [9]. Moreover, in quantum optics the question of the coherent states over finite spaces was studied [10, 11]. Formulations of the finite-dimensional quantum mechanics have been given in several papers [12–14, 3, 4].

In section 2 we study the basic relations of quantum mechanics on discrete finite space, in section 3 the coherent states are constructed and in section 4 some examples of overcomplete families of coherent states are presented.

## 2. Finite-dimensional quantum mechanics

The basic relations of finite-dimensional quantum mechanics will be presented following [12]. For the sake of simplicity we shall restrict our attention to one classical degree of freedom, i.e. the configuration space is the cyclic group $Z_M$, $M$ being a power of a prime [3]. Quantum theories for several degrees of freedom can be obtained as tensor products of theories of one degree.

Let $j$ take one of $M$ discrete values, $j = 0, 1, \ldots, M-1$. With each value of $j$, a vector $|j\rangle$ of an orthonormal basis of $M$-dimensional complex Hilbert space, $\mathcal{H}_M$, is associated. Then a *position operator* [12, 3, 4] is defined by

$$\hat{Q} = \sum_{j=0}^{M-1} j|j\rangle\langle j|.$$

The eigenvectors of $\hat{Q}$ form a basis $\{|j\rangle\}$ of the Hilbert space $\mathcal{H}_M$, and $j$ are the corresponding eigenvalues. We have $(|j\rangle)_i = \delta_{i,j}$ in this basis.

*Momentum operators* are defined via unitary shift operators on the set $\{|j\rangle\}$ closed into a periodic chain ($Z_M$ 'manifold') by the conditions of periodicity

$$|j\rangle = |j + M\rangle.$$

The one-step shift operator transforms the vectors $|j\rangle$,

$$\hat{U}(1) : |j\rangle \mapsto |j + 1\rangle \qquad (\text{modulo } M)$$

and its powers generate cyclic permutation matrices $\hat{U}(k) = (\hat{U}(1))^k$, $\hat{U}(k)|j\rangle = |j + k\rangle$, with $(\hat{U}(1))^M = \widehat{\mathrm{Id}}$. The operators $\hat{U}(k)$, $k = 0, 1, \ldots, M-1$, provide the regular representation of the cyclic group $Z_M$ in $\mathcal{H}$.

A *momentum operator*, $\hat{P}$, with eigenvalues, $k$, and eigenvectors, $|k\rangle$, is then defined in a similar way as in the continuous case [12], i.e. via

$$\hat{U}(1) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{M}\hat{P}}$$

in analogy with a generator of a one-parameter group of unitary transformations. The eigenvectors $|k\rangle$ of $\hat{P}$ corresponding to eigenvalues $k = 0, \ldots, M-1$, which can be expanded in position eigenvectors

$$|k\rangle = \frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}kj}|j\rangle. \tag{1}$$

Let us note that (1) is the discrete Fourier transformation of the eigenvectors $|j\rangle$. Matrix elements of the operator $\hat{P}$ in the basis of eigenvectors of $\hat{Q}$ are

$$\langle m|\hat{P}|n\rangle = \frac{1}{M}\sum_k k\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}k(m-n)} = \begin{cases} \dfrac{M-1}{2} & \text{if } m = n \\ \dfrac{1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}(m-n)} - 1} + \dfrac{1}{M} & \text{otherwise} \end{cases}$$

and matrix elements of the commutator are

$$\langle m|[\hat{Q}, \hat{P}]|n\rangle = (m-n)\langle m|\hat{P}|n\rangle.$$

Although these commutation relations differ from the continuous case, the exponential *Weyl relations* still hold in the discrete case [12] in the form [12, 3]

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}t\hat{Q}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}s\hat{P}} = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{M}ts}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}s\hat{P}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}t\hat{Q}} \tag{2}$$

or

$$\mathrm{e}^{\frac{\pi\mathrm{i}}{M}ts}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}t\hat{Q}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}s\hat{P}} = \mathrm{e}^{-\frac{\pi\mathrm{i}}{M}ts}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}s\hat{P}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}t\hat{Q}}.$$

The unitary operators $\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{Q}}$ and $\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}}$ generate the *finite Weyl group*, which acts irreducibly in the Hilbert space $\mathcal{H}_M$ [3]. In section 3 we shall make use of the expressions

$$\mathrm{e}^{\frac{\pi}{M}}\mathrm{e}^{\frac{2\pi}{M}\hat{Q}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}} = \mathrm{e}^{-\frac{\pi}{M}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}}\mathrm{e}^{\frac{2\pi}{M}\hat{Q}} \tag{3}$$

for special values $s = 1,\ t = -\mathrm{i}$, analogous to

$$\mathrm{e}^{-\frac{1}{2\hbar}}\mathrm{e}^{\frac{1}{\hbar}\hat{q}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{p}} = \mathrm{e}^{\frac{1}{2\hbar}}\mathrm{e}^{\frac{i}{\hbar}\hat{p}}\mathrm{e}^{\frac{1}{\hbar}\hat{q}} = \mathrm{e}^{\frac{1}{\hbar}(\hat{q}+\mathrm{i}\hat{p})}. \tag{4}$$

in the continuous case.

## 3. Construction of the coherent states

Let us first look for a 'vacuum' vector $|0,0\rangle$ in Hilbert space, $\mathcal{H}_M$, in analogy with the continuous case. In the continuous case of the usual harmonic oscillator the vacuum vector $|0\rangle_c$ is defined by $\hat{a}|0\rangle_c = 0$, or $\mathrm{e}^{\hat{a}}|0\rangle_c = |0\rangle_c$, where $\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{q} + \mathrm{i}\hat{p})$ is the annihilation operator. Making use of the analogy of (4) and (3), we look for a vector $|\phi\rangle$ in Hilbert space, $\mathcal{H}_M$, satisfying the relation

$$\mathrm{e}^{-\frac{\pi}{M}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}}\mathrm{e}^{\frac{2\pi}{M}\hat{Q}}|\phi\rangle = |\phi\rangle. \tag{5}$$

In order to solve (5), we expand $|\phi\rangle$ into a linear combination of eigenvectors $|j\rangle$ of position operator

$$|\phi\rangle = A_M \sum_{j=0}^{M-1} f_j|j\rangle \tag{6}$$

now if we apply (3) and obtain relations to be satisfied by the expansion coefficients $f_j$,

$$f_{j-1} = \mathrm{e}^{\frac{-\pi}{M}}\mathrm{e}^{\frac{2\pi j}{M}} f_j \qquad \text{for } j = 1, 2\ldots, M-1 \tag{7}$$

$$f_{M-1} = \mathrm{e}^{\frac{-\pi}{M}} f_0. \tag{8}$$

It turns out, however, that the 'periodicity' relation (8) for $j = 0$ cannot be matched with the recurrence relations (7), when $M = 3, 4, \ldots$, and the vector $|\phi\rangle$ *does not exist*. For $M = 2$, the solutions *do exist*: $f_0 = c,\ f_1 = c\mathrm{e}^{-\frac{\pi}{2}}, c =$ constant. Also in the limit $M \to \infty$ (i.e. discarding (8)) the solutions of (7) exist,

$$f_j = c\mathrm{e}^{-\frac{\pi}{M}j^2}.$$

A family of generalized coherent states of type $\Gamma(g), |\psi_0\rangle$ in the sense of Perelomov [15] is defined in a representation $\Gamma(g)$ of a group $G$ as a family of states $\{|\psi_g\rangle\}$, $|\psi_g\rangle = \Gamma(g)|\psi_0\rangle$, where $g$ runs over the whole group $G$ and $|\psi_0\rangle$ is the 'vacuum' vector.

In our case we try to follow the continuous case of the usual harmonic oscillator as a guide, with the coherent states

$$|x + \mathrm{i}y\rangle_c = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(y\hat{q}-x\hat{p})}|0\rangle_c \qquad \text{where } (x, y) \in \mathbb{R}^2$$

defined by the action of the Heisenberg–Weyl group representation on $|0\rangle_c \in L^2(\mathbb{R})$. So we will act in an analogous way on a 'vacuum' vector $|0,0\rangle$ by the representation (2) of the finite Weyl group and so define the *generalized coherent states of type* $\{\hat{W}(m,a), |\psi_0\rangle\}$, where $|\phi\rangle \in \mathcal{H}_M$ is arbitrary. The operators

$$\hat{W}(m,a) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{M}m\hat{P}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}a\hat{Q}}$$

are called *Weyl operators*. The coherent states $|m, a\rangle$ are thus defined by

$$|m, a\rangle = \hat{W}(m, a)|0, 0\rangle$$

or

$$|m, a\rangle = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{M}m\hat{P}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}a\hat{Q}}|0, 0\rangle \qquad m, a = 0, \ldots, M-1. \tag{9}$$

As the first choice of our 'vacuum' vector $|0.0\rangle$ we will take the vector with the coordinates $f_j = \mathrm{e}^{-\frac{\pi}{M}j^2}$, $j = 0, \ldots, M-1$, i.e. $|0, 0\rangle = A_M \sum_{j=0}^{M-1} f_j |j\rangle$, even if (5) is not satisfied. The normalization of $|0, 0\rangle$ requires

$$A_M = \left(\sum_{j=0}^{M-1} f_j^2\right)^{-\frac{1}{2}}.$$

Using the decomposition (6), the coherent states (9) can be expanded in terms of the position eigenvectors

$$|m, a\rangle = A_M \sum_{j=0}^{M-1} \mathrm{e}^{-\frac{\pi}{M}j^2}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}aj}|j + m\rangle.$$

For the inner product (overlap) of two coherent states in $\mathcal{H}$ the formula

$$\langle k, b|m, a\rangle = A_M^2 \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}(ak-bm)} \sum_{\rho=0}^{M-1} f_{\rho+k} f_{\rho+m} \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\rho(a-b)}$$

is obtained. With its help the following identity can be proved,

$$\sum_{k=0}^{M-1}\sum_{b=0}^{M-1} |k, b\rangle\langle k, b|m, a\rangle = M|m, a\rangle$$

which means that our coherent states form an *overcomplete system* in the usual sense [2]:

$$\frac{1}{M}\sum_{k=0}^{M-1}\sum_{b=0}^{M-1} |k, b\rangle\langle k, b| = \hat{\mathbf{1}}.$$

If the system is prepared in the coherent state $|k, b\rangle$, then the probability to measure the eigenvalue $j$ of position operator $\hat{Q}$, $\hat{Q}|j\rangle = j|j\rangle$ is

$$|\langle j|k, a\rangle|^2 = A_M^2 \mathrm{e}^{-\frac{\pi}{M}(j-k)^2}$$

and it is maximal when $j = k$. Similarly the probability to measure the eigenvalue $p$ of momentum operator $\hat{P}$, $\hat{P}|p\rangle = p|p\rangle$ is

$$|\langle p|k, a\rangle|^2 = |\frac{A_M}{M}\sum_{\rho} \mathrm{e}^{-\frac{\pi}{M}\rho^2}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\rho(p-a)}|^2$$

and it is maximal when $p = a$.

Denoting $z = m + \mathrm{i}a$, $|z\rangle = |m, a\rangle$, where $m, a \in Z_M$, one can simply show that in the limit $M \to \infty$

$$\mathrm{e}^{-\frac{\pi}{M}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}}\mathrm{e}^{\frac{2\pi}{M}\hat{Q}}|z\rangle = \mathrm{e}^{\frac{2\pi}{M}z}|z\rangle \tag{10}$$

and, of course, the expectation value of $\mathrm{e}^{\frac{\pi}{M}}\mathrm{e}^{\frac{2\pi}{M}\hat{Q}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}}$ is

$$\langle z|\mathrm{e}^{\frac{\pi}{M}}\mathrm{e}^{\frac{2\pi}{M}\hat{Q}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}}|z\rangle = \mathrm{e}^{\frac{2\pi}{M}z}$$

Let us denote $x = l + \mathrm{i}c,\ y = k + \mathrm{i}b$ and $w = n + \mathrm{i}d$, where $l, c, k, b, n, d \in Z_M$. Then our coherent states clearly define a *reproducing kernel* $K(x, y)$,

$$K(x, y) = \frac{1}{A_M M}\langle x|y\rangle$$

with the reproducing property

$$K(x, y) = \sum_n \sum_d K(x, w)\, K(w, y).$$

*Remark 1.* Modifying the prefactor $\mathrm{e}^{\frac{-\pi}{M}}$ in condition (5) to the form

$$\mathrm{e}^{\beta}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}\hat{P}}\mathrm{e}^{\frac{2\pi}{M}\hat{Q}}|0, 0\rangle_s = |0, 0\rangle_s \tag{11}$$

one can find $\beta$ and a new 'vacuum' vector $|0, 0\rangle_s$ for any $M = 2, 3, \ldots$. Namely, let $|0, 0\rangle_s$ be expanded in position eigenvectors $|j\rangle$,

$$|0, 0\rangle_s = B_M \sum_{j=0}^{M-1} g_j |j\rangle. \tag{12}$$

Substituting (12) into (11) and using periodicity conditions one can show that

$$\beta = -\pi \frac{M-1}{M} \qquad \text{and} \qquad g_j = \mathrm{e}^{-\frac{\pi}{M}j^2 - \frac{2\pi}{M}j + \pi j}.$$

Applying the same procedure as in (9) on the new 'vacuum' $|0, 0\rangle_s$, one can define a new family of *coherent states* $|m, a\rangle_s$ by

$$|m, a\rangle_s = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{M}m\hat{P}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}a\hat{Q}}|0, 0\rangle_s \qquad m, a = 0, \ldots, M-1.$$

However, the expectation values for the coherent states $|k, m\rangle_s$ differ from the expectation values for $|k, m\rangle$. If the system is prepared in the coherent state $|k, b\rangle_s$, then the probabillity to measure the eigenvalue $j$ of position operator $\hat{Q}$, $\hat{Q}|j\rangle = j|j\rangle$ is

$$|\langle j|k, a\rangle_s| = B_M \mathrm{e}^{-\frac{\pi}{M}(j-k)^2 - \frac{2\pi}{M}(j-k) - \pi(j-k)}$$

and it is maximal when $j = k - 1 + \frac{M}{2}$.

*Remark 2.* The overcompleteness property can be traced back to the proposition in [16]. The set of $M^2$ Weyl operators $\hat{W}(m, a)$ satisfies

$$\mathrm{tr}(\hat{W}(m, a)\hat{W}(m, a)^*) = M\delta_{ss'}\delta_{tt'}$$
$$\sum_{m,a}\hat{W}(m, a)\hat{Y}\hat{W}(m, a)^* = M\hat{I}\,\mathrm{tr}\,\hat{Y}$$

where $\hat{Y}$ is an arbitrary complex $M \times M$ matrix.

Namely, applying the second identity in Schwinger's proposition to the projection operator $\hat{Y} = |\psi_0\rangle\langle\psi_0|$, with $\mathrm{tr}\,\hat{Y} = 1$, the *overcompleteness* of generalized coherent states of type $\{\hat{W}(m, a), |\psi_0\rangle\}$ is a straightforward consequence:

$$\sum_{m,a}|m, a\rangle\langle m, a| = M\hat{I}.$$

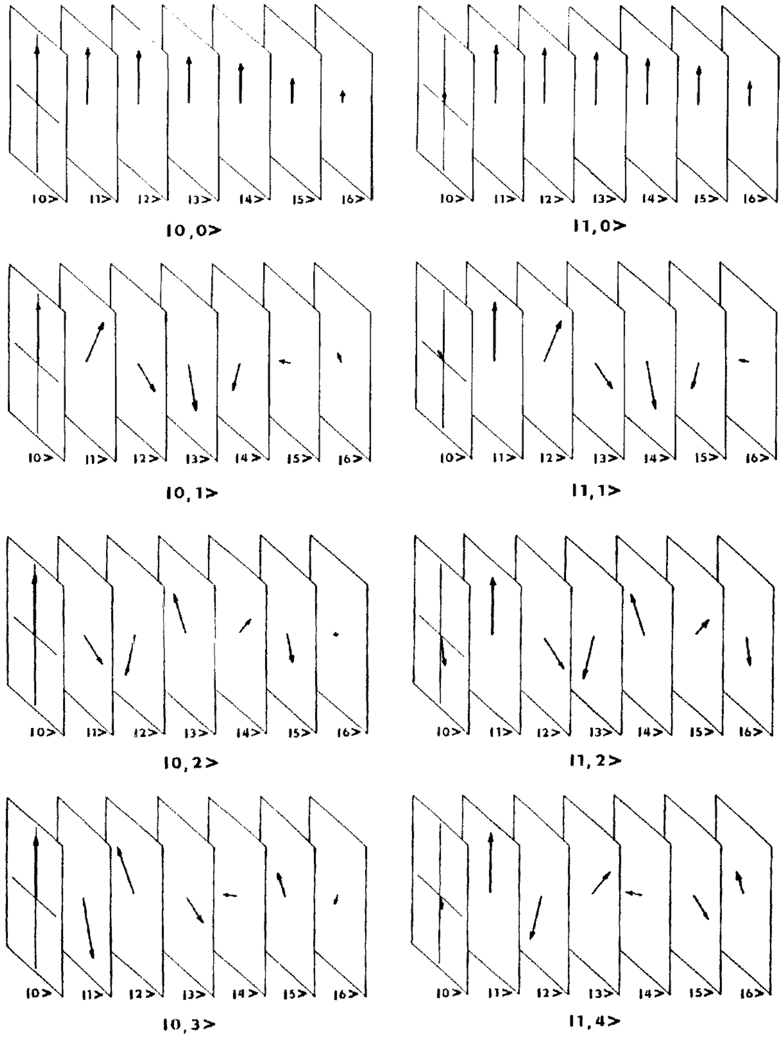


**Figure 1.** Phase space $Z_7 \times Z_7$. The planes correspond to complex eigenvectors of position operators, where the vertical axis is imaginary and the horizontal axis is real. The arrows then show the projections of the given coherent state on each position eigenvector. Since the lengths of projections $l$ on different complex planes differ by several orders, we have modified them: the length of drawn arrows is $\ln(\frac{l}{e^{-20}})$.

*Remark 3.* As is well known, coherent states have been at the heart of quantum optics from its beginnings. They are the quantum states which seem the closest, one can approach a state of classical electromagnetic fields of well defined amplitude and phase. It is remarkable that recently, in order to properly define a *phase operator* canonically conjugate to the number operator, the formalism of quantum mechanics in Hilbert space of finite dimension was used (see, e.g. [10], and references therein). The eigenstates of these operators are our bases $|j\rangle$ and $|k\rangle$, related by the discrete Fourier transformation (1). Thus, $\hat{Q}$ is interpreted as the number operator and its eigenstates $|j\rangle$ as the first $M$ bosonic Fock states. The phase $\theta_k$ is

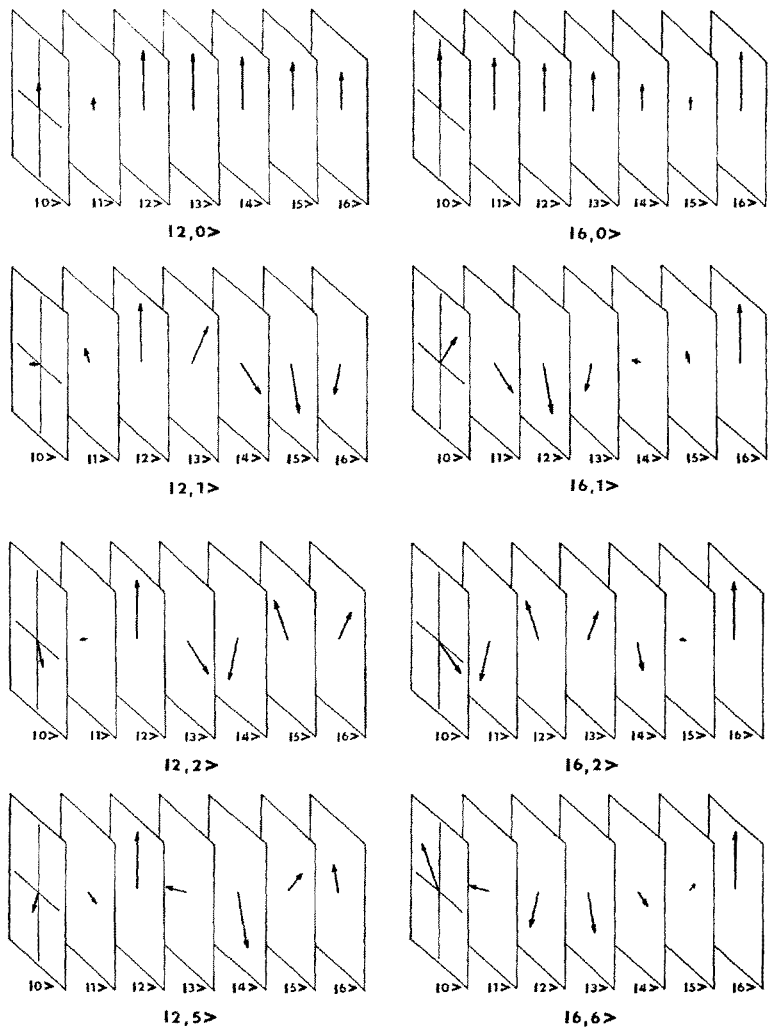


**Figure 1.** (Continued)

connected with our momentum $k$ by

$$\theta_k = \theta_0 + 2\pi \frac{k}{M}$$

where $\theta_0$ is chosen arbitrary, defining a phase window. After evaluating physical quantities in $\mathcal{H}_M$, the limit $M \to \infty$ is taken.

## 4. Examples

In this section we give explicit formulae for the families of coherent states with $M = 2, 3, 4$ and also drawings of projections for eight coherent states with $M = 7$ (figure 1).

Phase space $Z_2 \times Z_2$:

$$|0, 0\rangle = \frac{1}{\sqrt{1 + \mathrm{e}^{-\pi}}} (|0\rangle + \mathrm{e}^{-\frac{\pi}{2}} |1\rangle)$$

$$|0,1\rangle = \frac{1}{\sqrt{1+\mathrm{e}^{-\pi}}}(|0\rangle - \mathrm{e}^{-\frac{\pi}{2}}|1\rangle)$$
$$|1,0\rangle = \frac{1}{\sqrt{1+\mathrm{e}^{-\pi}}}(|1\rangle + \mathrm{e}^{-\frac{\pi}{2}}|0\rangle)$$
$$|1,1\rangle = \frac{1}{\sqrt{1+\mathrm{e}^{-\pi}}}(|1\rangle - \mathrm{e}^{-\frac{\pi}{2}}|0\rangle).$$

Phase space $Z_3 \times Z_3$:

$$|0,0\rangle = A_3(|0\rangle + \mathrm{e}^{-\frac{\pi}{3}}|1\rangle + \mathrm{e}^{-\frac{4\pi}{3}}|2\rangle)$$
$$|0,1\rangle = A_3(|0\rangle + \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\pi}{3}}|1\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{4\pi}{3}}|2\rangle)$$
$$|0,2\rangle = A_3(|0\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\pi}{3}}|1\rangle + \mathrm{e}^{\frac{8\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{4\pi}{3}}|2\rangle)$$
$$|1,0\rangle = A_3(|1\rangle + \mathrm{e}^{-\frac{\pi}{3}}|2\rangle + \mathrm{e}^{-\frac{4\pi}{3}}|0\rangle)$$
$$|1,1\rangle = A_3(|1\rangle + \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\pi}{3}}|2\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{4\pi}{3}}|0\rangle)$$
$$|1,2\rangle = A_3(|1\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\pi}{3}}|2\rangle + \mathrm{e}^{\frac{8\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{4\pi}{3}}|0\rangle)$$
$$|2,0\rangle = A_3(|2\rangle + \mathrm{e}^{-\frac{\pi}{3}}|0\rangle + \mathrm{e}^{-\frac{4\pi}{3}}|1\rangle)$$
$$|2,1\rangle = A_3(|2\rangle + \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\pi}{3}}|0\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{4\pi}{3}}|1\rangle)$$
$$|2,2\rangle = A_3(|2\rangle + \mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}\mathrm{e}^{-\frac{\pi}{3}}|0\rangle + \mathrm{e}^{\frac{8\pi\mathrm{i}}{3}}e^{-\frac{4\pi}{3}}|1\rangle)$$

where $A_3 = \frac{1}{\sqrt{1+\mathrm{e}^{-\frac{2\pi}{3}}+\mathrm{e}^{-\frac{8\pi}{3}}}}$.

Phase space $Z_4 \times Z_4$:

$$|0,0\rangle = A_4(|0\rangle + \mathrm{e}^{-\frac{\pi}{4}}|1\rangle + \mathrm{e}^{-\pi}|2\rangle + \mathrm{e}^{-\frac{9\pi}{4}}|3\rangle)$$
$$|0,1\rangle = A_4(|0\rangle + \mathrm{i}\mathrm{e}^{-\frac{\pi}{4}}|1\rangle - \mathrm{e}^{-\pi}|2\rangle - \mathrm{i}\mathrm{e}^{-\frac{9\pi}{4}}|3\rangle)$$
$$|0,2\rangle = A_4(|0\rangle - \mathrm{e}^{-\frac{\pi}{4}}|1\rangle + \mathrm{e}^{-\pi}|2\rangle - \mathrm{e}^{-\frac{9\pi}{4}}|3\rangle)$$
$$|0,3\rangle = A_4(|0\rangle - \mathrm{i}\mathrm{e}^{-\frac{\pi}{4}}|1\rangle - \mathrm{e}^{-\pi}|2\rangle + \mathrm{i}\mathrm{e}^{-\frac{9\pi}{4}}|3\rangle)$$
$$|1,0\rangle = A_4(|1\rangle + \mathrm{e}^{-\frac{\pi}{4}}|2\rangle + \mathrm{e}^{-\pi}|3\rangle + \mathrm{e}^{-\frac{9\pi}{4}}|0\rangle)$$
$$|1,1\rangle = A_4(|1\rangle + \mathrm{i}\mathrm{e}^{-\frac{\pi}{4}}|2\rangle - \mathrm{e}^{-\pi}|3\rangle - \mathrm{i}\mathrm{e}^{-\frac{9\pi}{4}}|0\rangle)$$
$$|1,2\rangle = A_4(|1\rangle - \mathrm{e}^{-\frac{\pi}{4}}|2\rangle + \mathrm{e}^{-\pi}|3\rangle - \mathrm{e}^{-\frac{9\pi}{4}}|0\rangle)$$
$$|1,3\rangle = A_4(|1\rangle - \mathrm{i}\mathrm{e}^{-\frac{\pi}{4}}|2\rangle - \mathrm{e}^{-\pi}|3\rangle + \mathrm{i}\mathrm{e}^{-\frac{9\pi}{4}}|0\rangle)$$
$$|2,0\rangle = A_4(|2\rangle + \mathrm{e}^{-\frac{\pi}{4}}|3\rangle + \mathrm{e}^{-\pi}|0\rangle + \mathrm{e}^{-\frac{9\pi}{4}}|1\rangle)$$
$$|2,1\rangle = A_4(|2\rangle + \mathrm{i}\mathrm{e}^{-\frac{\pi}{4}}|3\rangle - \mathrm{e}^{-\pi}|0\rangle - \mathrm{i}\mathrm{e}^{-\frac{9\pi}{4}}|1\rangle)$$
$$|3,0\rangle = A_4(|3\rangle + \mathrm{e}^{-\frac{\pi}{4}}|0\rangle + \mathrm{e}^{-\pi}|1\rangle + \mathrm{e}^{-\frac{9\pi}{4}}|2\rangle)$$
$$|3,3\rangle = A_4(|3\rangle - \mathrm{i}\mathrm{e}^{-\frac{\pi}{4}}|0\rangle - \mathrm{e}^{-\pi}|1\rangle + \mathrm{i}\mathrm{e}^{-\frac{9\pi}{4}}|2\rangle)$$

where $A_4 = \frac{1}{\sqrt{1+\mathrm{e}^{-\frac{\pi}{2}}+\mathrm{e}^{-2\pi}+\mathrm{e}^{-\frac{9\pi}{2}}}}$.

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